

# Classical transverse Ising spin glass with short- range interaction beyond the mean field approximation

K Walasek, K Lukierska- Walasek

Institute of Physics, Pedagogical University, Plac Słowiański 6, 65- 069 Zielona Góra, Poland

and

L De Cesare, I Rabuffo

Dipartimento di Scienze Fisiche "E. R. Caianiello" Università di Salerno, 84100 Salerno, and

INFM Unità di Salerno, Italy

## Abstract

The classical transverse field Ising spin- glass model with short-range interactions is investigated beyond the mean- field approximation for a real  $d$ - dimensional lattice. We use an appropriate nontrivial modification of the Bethe- Peierls method recently formulated for the Ising spin- glass. The zero- temperature critical value of the transverse field and the linear susceptibility in the paramagnetic phase are obtained analytically as functions of dimensionality  $d$ . The phase diagram is also calculated numerically for different values of  $d$ . In the limit  $d \rightarrow \infty$ , known mean- field results are consistently reproduced.

75.10.Nr, 75.50.Lk

The study of glasses is today one of the most relevant and actual problem in condensed matter physics. Originally, the basic idea was to start from spin glass (SG) models and to extract as much as it was possible at a mean- field- approximation (MFA) level [1–4]. However, there are recent studies [5–8] which indicate difficulties to extend the MFA scenario to realistic spin glasses with short- range interaction and decide ”a priori” which properties survive and which must be appropriately modified. Renormalization group treatments [9–11] for classical and quantum spin glasses and phenomenological studies [12] do not seem to suggest a clear picture.

Quite recently, in an interesting paper [13], an approach beyond the MFA has been achieved for an  $d$ - dimensional Ising SG model with short- range interactions on a real lattice using an extension of the Bethe- Peierls approximation (BPA) [14] to the spin glass problem via the replica trick. This approach seems to be very promising to establish a direct contact with the results obtained by different authors for the infinite- ranged version and to controll possible deviations for short- ranged glasses from the well acquired MFA scenario. Of course, additional applications to more complex glassy systems and improvements are necessary for understanding something more about the role played by the glassy fluctuations around the MFA solution for finite dimensionalities.

In this paper we explore the glassy properties of the  $d$ - dimensional classical transverse field Ising SG model [15–17] with short-range interactions using an appropriate nontrivial modifications of the BPA formulated originally in Ref. [13] for the Ising SG. The model here considered has received recent attention because it is a relatively simple SG model which reflects some properties of the quantum counterpart [4] and it is a specific example of classical two- vector anisotropic SG. So, the classical limit of the usual more complex quantum Ising SG model in a transverse field with realistic exchange interactions may be useful for taking contact [16,17] with low- temperature properties of the so- called ”proton glasses” [15], as the compounds  $\text{Rb}_{1-x}(\text{NH}_4)_x\text{H}_2\text{PO}_4$ , and with the most recent experimental magnetic data for the dipolar glass  $\text{Li Ho}_{0.167}\text{Y}_{0.822}\text{F}_4$  [18,19].

As concerning the quantum realistic SG model, a lot of results have been obtained only

for  $d = 1$  [20,21] and for infinite- ranged interactions ( $d = \infty$ ) [4,19]. For the classical counterpart, only results with infinite- ranged interaction have been derived [16,17], in particular at  $T = 0$ , and even in the "simple" MFA limit the full phase diagram has not yet been calculated. In any case, there are rather little studies about short- ranged glassy models. So, it appears quite relevant that the BPA allows us to describe some nontrivial glassy properties of the model (in the paramagnetic phase) for  $d \geq 1$ . In particular, explicit analytical results are obtained at  $T = 0$  and the phase diagram in the (temperature, transverse field)- plane is derived numerically for an arbitrary dimensionality.

The classical transverse field SG model here considered is described by the Hamiltonian [15–17]:

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} S_i^z S_j^z - \Gamma \sum_{i=1}^N S_i^x - h \sum_{i=1}^N S_i^z \quad (1)$$

with  $(S_i^z)^2 + (S_i^x)^2 = 1$ . Here  $\Gamma$  and  $h$  are transverse and longitudinal fields, respectively, the couplings  $J_{ij}$ 's are independent random variables assuming values  $\pm J$  with the equal probability. In (1)  $\sum_{\langle i,j \rangle} \dots$  denotes a sum over nearest- neighbours pairs of  $N$  sites on a hypercubic  $d$ - dimensional lattice. Using the replica trick, all is reduced to determine the "quenched average":

$$Z_n = \left[ \text{Tr} \exp \left( -\beta \sum_{\alpha=1}^n H_{\alpha} \right) \right]_{\text{av}}, \quad (2)$$

where  $H_{\alpha}$  is the  $\alpha$ -th replica of the Hamiltonian (1) and  $\beta = 1/T$  with the Boltzmann constant  $k_B \equiv 1$ . Working directly on the real lattice, the basic idea of the BPA for spin glasses [13] is to take into account the correct interactions inside replicated clusters (cl), constituted by a central spin  $\mathbf{S}_0$  and its  $2d$  nearest- neighbours  $\{\mathbf{S}_i; i = 1, \dots, 2d\}$ , and to describe the interactions of the cluster borders with the remnant (rm) of the system by means of effective couplings among replicas to be determined self - consistently. With this in mind, Eq.(2) for the Bethe- Peierls ansatz can be formally rewritten as [13]:

$$\begin{aligned} Z_n &= \text{Tr}_{\{\mathbf{S}_{\text{cl}}\}} \left[ \exp \left( -\beta \sum_{\alpha=1}^n H_{\alpha}^{(\text{cl})} \right) \text{Tr}_{\{\mathbf{S}_{\text{rm}}\}} \exp \left( -\beta \sum_{\alpha=1}^n H_{\alpha}^{(\text{rm})} \right) \right]_{\text{av}} \\ &\equiv K(T, \Gamma, h) \text{Tr}_{\{\mathbf{S}_{\text{cl}}\}} [\exp(-\beta \mathcal{H}_n)]_{\text{av}} \end{aligned} \quad (3)$$

where

$$H_{\alpha}^{\text{cl}} = -\Gamma \sum_{k=0}^{2d} S_{k\alpha}^x - \sum_{i=1}^{2d} J_{0i} S_{0\alpha}^z S_{i\alpha}^z \quad (4)$$

and  $H_{\alpha}^{(\text{rm})}$  denotes replicated Hamiltonians of the cluster and remnant of the system interacting with cluster borders, respectively, and  $K(T, \Gamma, h)$  is a multiplicative constant independent on lateral spins,

$$\text{Tr}_{\{\mathbf{s}_{\text{cl}}\}} \cdots = \frac{1}{\pi^{n(2d+1)}} \int_{-1}^1 \prod_{k=0}^{2d} \prod_{\alpha=1}^n \frac{dS_{k\alpha}^z}{\sqrt{1 - (S_{k\alpha}^z)^2}} \cdots \quad (5)$$

and

$$\begin{aligned} \mathcal{H}_n = & - \sum_{\alpha=1}^n \sum_{i=1}^{2d} J_{0i} S_{0\alpha}^z S_{i\alpha}^z - \frac{\beta J^2}{2} \sum_{\alpha, \alpha'=1}^n \sum_{i=1}^{2d} \lambda_{\alpha\alpha'} S_{i\alpha}^z S_{i\alpha'}^z \\ & - \frac{1}{\beta} \sum_{\alpha=1}^n \sum_{k=0}^{2d} \ln \cosh \left[ \beta \Gamma \sqrt{1 - (S_{k\alpha}^z)^2} \right] - h \sum_{\alpha=1}^n \sum_{i=1}^{2d} S_{k\alpha}^z \end{aligned} \quad (6)$$

with  $\lambda_{\alpha\alpha'} = \mu_{\alpha\alpha'}$  for  $\alpha \neq \alpha'$  and  $\lambda_{\alpha\alpha} = \mu$  which are parameters to be determined via appropriate self-consistent equations. Here we have used the relation  $S_{k\alpha}^x = \pm \sqrt{1 - (S_{k\alpha}^z)^2}$  ( $k = 0, 1, \dots, 2d$ ). Of course, if a transition from a paramagnetic phase to a SG one is assumed to exist, one expects  $\mu_{\alpha\alpha'} = 0$  in the paramagnetic phase.

At this stage, the self-consistent equations which determine the effective couplings  $\mu_{\alpha, \alpha'}$  and  $\mu$  as  $n \rightarrow 0$ , are

$$\langle S_{i\alpha}^z S_{i\alpha'}^z \rangle = \langle S_{0\alpha}^z S_{0\alpha'}^z \rangle \quad \text{with } i = 1, \dots, 2d, \quad (7)$$

where

$$\langle \cdots \rangle = \frac{\text{Tr} [\exp(-\beta \mathcal{H}_n) \cdots]_{\text{av}}}{\text{Tr} [\exp(-\beta \mathcal{H}_n)]_{\text{av}}} . \quad (8)$$

It is easy to check that, for  $h = 0$ , due to the inversion symmetry  $S_{i\alpha}^z \rightarrow -S_{i\alpha}^z$  and symmetry of the probability distribution for  $J_{ij}$  Eq. (7) with  $\alpha = \alpha'$  can be reduced to the following one:

$$\chi_i = \chi_0 \quad \text{for } i = 1, \dots, 2d, \quad (9)$$

where

$$\chi_k = \frac{\partial \langle S_{k\alpha} \rangle}{\partial h} \Big|_{h=0} \text{ with } k = 0, 1, \dots, 2d \quad (10)$$

denotes the local susceptibility.

We are now in the position to obtain the explicit equations for  $\mu$  which will be used for obtaining also the phase diagram of the model. Since it is expected that  $\mu_{\alpha, \alpha'} \rightarrow 0$  approaching the spin- glass transition from below, for  $n \rightarrow 0$ , one obtains at  $h = 0$

$$\begin{aligned} \langle S_{k\alpha}^z S_{k\alpha'}^z \rangle &= \beta J^2 \mu_{\alpha\alpha'} \sum_{i=1}^{2d} [\langle S_k^z S_i^z \rangle_0]_{\text{av}} + \mathcal{O}(\mu_{\alpha, \alpha'}^2) \\ (k &= 0, 1, \dots, 2d) , \end{aligned} \quad (11)$$

where

$$\langle \dots \rangle_0 = \frac{1}{\pi^{2d+1} Z_0} \int_{-1}^1 \prod_{k=0}^{2d} \frac{dS_k^z}{\sqrt{1 - (S_k^z)^2}} \exp(-\beta \mathcal{H}_0) \dots \quad (12)$$

with

$$\mathcal{H}_0 = - \sum_{i=1}^{2d} J_{0i} S_0^z S_i^z - \frac{\beta J^2}{2} \mu \sum_{i=1}^{2d} (S_i^z)^2 - \frac{1}{\beta} \sum_{k=0}^{2d} \ln \cosh \left[ \beta \Gamma \sqrt{1 - (S_k^z)^2} \right] . \quad (13)$$

In Eq. (12)  $Z_0$  denotes the normalization factor. The term  $-h \sum_{k=0}^{2d} S_k^z$  must be added to the right hand side of (13) when it is necessary. So, at  $h = 0$  due to the translational symmetry for the sample averaged system and assuming  $\mu_{\alpha\alpha'} = 0$  at and above the glassy transition line (to be determined), the self- consistent equation (7) for  $\alpha \neq \alpha'$  and  $\alpha = \alpha'$  reduces ,respectively, to:

$$\begin{aligned} \left[ \langle (S_i^z)^2 \rangle_0^2 \right]_{\text{av}} + (2d - 1) \left[ \langle S_i^z S_j^z \rangle_0^2 \right]_{\text{av}} &= 2d \left[ \langle S_0^z S_j^z \rangle_0^2 \right]_{\text{av}} \\ (i &\neq j = 1, \dots, 2d) \end{aligned} \quad (14)$$

and

$$\left[ \langle (S_i^z)^2 \rangle_0 \right]_{\text{av}} = \left[ \langle (S_0^z)^2 \rangle_0 \right]_{\text{av}} \quad (i = 1, \dots, 2d) , \quad (15)$$

where  $i \neq j$  denote arbitrary lateral sites of the cluster with the central spin  $\mathbf{S}_0$ .

By solving Eqs (14)- (15), it is possible to obtain the phase diagram of our model in the  $(T, \Gamma)$  plane. Explicit results can be derived analytically only at  $T = 0$ . As  $T \rightarrow 0$ , introducing  $\bar{\mu} = \beta\mu$  which is finite, with the help of Eq. (12) choosing in Eq. (15)  $i = 1$  one obtains:

$$\bar{\mu} = \frac{1}{J^2} \frac{\Gamma - \sqrt{\Gamma^2 - 4(2d-1)J^2}}{2} \quad (16)$$

and hence taking into account Eq. (9) one finds, in the paramagnetic phase at  $T = 0$ , linear susceptibility:

$$\chi = 2 \frac{\Gamma + \sqrt{\Gamma^2 - 4(2d-1)J^2}}{\left(\Gamma + \sqrt{\Gamma^2 - 4(2d-1)J^2}\right)^2 - 4J^2} \quad (17)$$

Now, we calculate the critical value  $\Gamma_c$  at  $T = 0$  of the transverse field using Eqs (14) and (16). For  $i = 1$  and  $j \neq 1$ , with some algebra we rewrite Eq.(14), as  $T \rightarrow 0$ , in the following form:

$$\left(\Gamma_c - \bar{\mu}_c J^2\right)^4 - 2dJ^2 \left(\Gamma_c - \bar{\mu}_c J^2\right)^2 + (2d-1)J^4 = 0 \quad (18)$$

with  $\bar{\mu}_c J^2 = \frac{1}{2} \left[\Gamma_c - \sqrt{\Gamma_c^2 - 4(2d-1)J^2}\right]$ . From this equation one easily obtains:

$$\Gamma_c = 2(2d-1)^{1/2} J \quad (19)$$

As we see,  $\chi$  is positive and has physical meaning only for  $\Gamma \geq \Gamma_c$ . This suggests that the expression (17) for  $\chi$  is related only to the paramagnetic phase.

For  $(\Gamma - \Gamma_c)/\Gamma_c \ll 1$  Eq.(17) yields:

$$\chi \approx \begin{cases} \frac{(2d-1)^{1/2}}{2J(d-1)} \left[1 - \frac{d\sqrt{2}}{d-1} \left(\frac{\Gamma-\Gamma_c}{\Gamma_c}\right)^{1/2} + \mathcal{O}\left(\frac{\Gamma-\Gamma_c}{\Gamma_c}\right)\right] & \text{for } d \neq 1 \\ \frac{1}{2J\sqrt{2}} \left(\frac{\Gamma-\Gamma_c}{\Gamma_c}\right)^{-1/2} \left[1 + \mathcal{O}\left(\frac{\Gamma-\Gamma_c}{\Gamma_c}\right)^{1/2}\right] & \text{for } d = 1 \end{cases} \quad (20)$$

. As an "a posteriori" justification of the correctness of the glassy BPA (3), it is easy to check analytically that, using the rescaling  $J \rightarrow J/\sqrt{2d}$  one finds  $\bar{\mu} = 2d\chi$  and we get at  $T = 0$  for  $d \rightarrow \infty$ ;  $\chi = \frac{1}{2J^2} \left[\Gamma - (\Gamma^2 - \Gamma_c^2)^{1/2}\right]$  with  $\Gamma_c = 2J + \mathcal{O}(d^{-1/2})$ . These results reproduce exactly those obtained at  $T = 0$  for the same SG model but with infinite- ranged interactions [16]. This partial result supports the validity of the BPA for SG's.

The situation for  $d = 1$  with a divergence of the linear susceptibility at  $T = 0$  as  $\Gamma \rightarrow \Gamma_c^+$  can be simply explained. With the dichotomic probability distribution of one - dimensional nearest- neighbours couplings  $J(i, i + 1) \equiv J_i$ , ( $J_i = \pm J$  with  $J > 0$ ) after the gauge transformation of spin variables

$$S_i^z \rightarrow \text{sign}(J_1) \cdots \text{sign}(J_{i-1}) S_i^z \quad (21)$$

the system can be reduced to the uniform ferromagnet in an external transverse field  $\Gamma$ . Therefore it is naturally to expect that at  $\Gamma = \Gamma_c$  the ferromagnetic phase transition with a divergent linear susceptibility occurs. Indeed a more detailed analysis of the one dimensional case shows that at  $T = 0$  the linear susceptibility  $\chi$  can be calculated exactly for the paramagnetic phase. The divergence of  $\chi$  is the same as that obtained within the BPA for the one- dimensional system.

For arbitrary  $T$  and  $\Gamma$ , from Eqs (14) and (15) one can calculate numerically equilibrium properties of our model in the paramagnetic phase. In particular in Fig.1, the phase diagram in the  $(\Gamma, T)$ - plane for different  $d$  is shown. We have conveniently scaled variables  $T$  and  $\Gamma$  for reproducing results at very high dimensionality. In Fig. 2 a variation of rescaled critical temperature with a dimension at  $\Gamma = 0$  is plotted.

In conclusion, we have studied some relevant aspects of the classical transverse field short-ranged Ising SG in the paramagnetic phase for arbitrary dimensionality  $d$ . We expect that our results may be also useful for explaining some properties of the quantum counterpart of the model here considered. However, some questions remain to be explained. For example, on the basis of the general self- consistent equation (7) it is interesting to find solutions with  $\mu_{\alpha\alpha'} \neq 0$  in order to see if the BPA is able to describe correctly our model in the SG phase at arbitrary dimensionalities. Within present calculations working for paramagnetic phase this is practically impossible, since the complicated integral (12) has been reduced at  $T = 0$  to the asymptotic form being Gaussian like one. Such an asymptotic form is insufficient when parameters  $\mu_{\alpha,\alpha'}$  are included even in the replica symmetric form. Therefore further works will be necessary to elucidate these problems.

Authors thank to Professor M. Fusco Girard for numerical calculations. Discussions with Professor Th. M. Nieuwenhuizen, Drs R. Monasson and R. Zecchina are appreciated. Two of us (K. W. and K. L- W.) would like to express our thanks for the Department of Theoretical Physics of Salerno University shown to us during the preparation of this paper. Additional support from Polish Committee for Scientific Research (K. B. N.), Grant No 2 P03B 034 11 is gratefully acknowledged.



## REFERENCES

- [1] M. Mezard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore , 1988) and references therein.
- [2] K. H. Fisher and J. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge 1991).
- [3] K. Binder and J. D. Reger, Adv. Phys. **41**,547 (1992) and references therein.
- [4] B. K. Chakrabarti, A. Dutta, and P.Sen, *Quantum Ising Phases and Transitions in Transverse Ising Models* (Springer- Verlag, Berlin 1996).
- [5] C. M. Newman and D. L. Stein, Phys. Rev. Lett. **76**, 515 (1996).
- [6] G. Parisi, preprint, cond- mat/ 9603101 (1996).
- [7] G. Parisi, preprint, cond- mat/ 9701034 (1997).
- [8] A. Khurana, Phys. Rev. B **54**, 7295 (1996).
- [9] C. De Dominicis and I. Kondor, Phys. Rev. B **27**, 606 (1983); M.- C. Chang, Solid State Comm. **47**, 529 (1983); P. Le Doussal and A. B. Harris, Phys. Rev. B **40**, 9249 (1989).
- [10] C. De Dominicis, I. Kondor, and T. Temesvary, J. Phys. A **24**, L301 (1991).
- [11] N. Read, S. Sachdev, and J. Ye, Phys. Rev. B **52**, 384 (1995); N. Read and S. Sachdev, Nucl. Phys. B (Proc. Suppl.) **45A**, 38 (1996).
- [12] D. S. Fisher and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986); J. Phys. A **20**, L1005 (1987); A. J. Bray, Comments Condens. Matter Phys. **14**, 21 (1988).
- [13] M. Serva and G. Paladin, Phys. Rev. E **54**, 4637 (1996).
- [14] H. Bethe, Proc. R. Soc. London **150**, 552 (1935); R. Peierls, Proc. R. Soc. London **154**, 207 (1936).
- [15] R. Pirc, B. Tadić, and R. Blinc, Z. Phys. B- Cond. Matt. **61**, 69 (1985).

- [16] P. Pirc, P. Prelovšek and B. Tadić, Solid State Comm. **64** 137 (1987).
- [17] M. S. Li, K. Walasek, and M. Cieplak, Phys. Rev. B **56**, 1 (1997).
- [18] W. Wu et al, Phys. Rev. Lett. **67**, 2076 (1991).
- [19] J. Miller and D. A. Huse, Phys. Rev. Lett. **70**, 3147 (1993).
- [20] R. Shankar and G. Murthy, Phys. Rev. B **36**, 336 (1987); B. Mc Coy and T. T. Wu, Phys. Rev. **176**, 631 (1968).
- [21] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992).

## FIGURES

FIG. 1. Phase diagram of the classical transverse Ising spin- glass with short- range interaction within the Bethe- Peierls approximation for spatial dimensions  $d = 2, 3, 4, 5$  and  $6$ . The temperature  $T$  and transverse field  $\Gamma$  are rescaled by the factor  $(2d)^{-1/2}$ . The larger the dimension, the higher the corresponding line. Here  $J \equiv 1$ .

FIG. 2. Rescaling critical temperature  $T_c (2d)^{-1/2}$  for  $\Gamma = 0$  versus the dimension  $d$ . Here  $J \equiv 1$ .

This figure "fig1.gif" is available in "gif" format from:

<http://arXiv.org/ps/cond-mat/9804033v1>

This figure "fig2.gif" is available in "gif" format from:

<http://arXiv.org/ps/cond-mat/9804033v1>